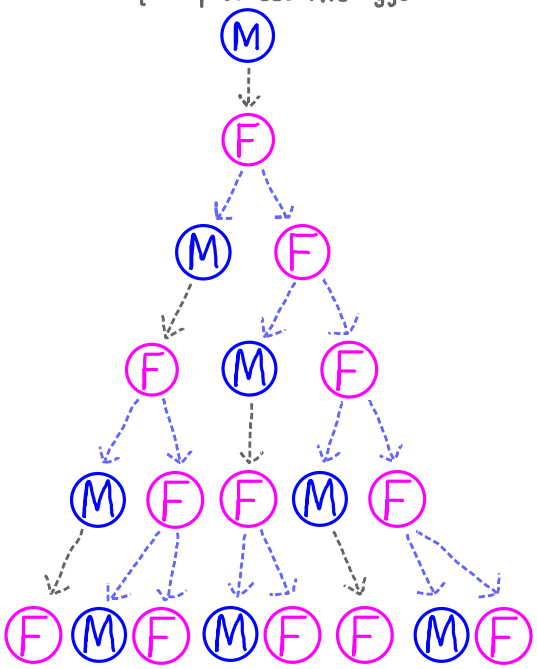
WMW#3: Playing With Honeycombs

Motivating Example
Reproduction of bees

FYI: A fertilized egg produced by a male produces a female. The queen produces two eggs.

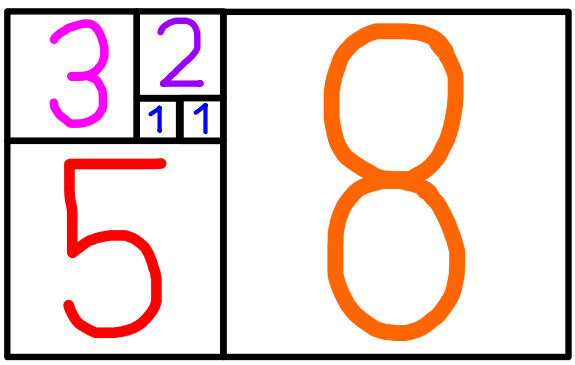


and reproduction continues!

In each generation that the bees reproduce, it takes the sum of the previous two generations of bees. As a result, it becomes a Fibonacci sequence.

Fibonacci numbers: [1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144]

Why Fibonacci numbers?



$1+1=2$
$1+2=3$
$2+3=5$
$3+5=8$

$5+8=13$
$8+13=21$
$13+21=34$
$21+34=55$

The recurrence relation for Fibonacci numbers is:

$$F_n = F_{n-1} + F_{n-2}$$

given the initial conditions $F_0 = 1$ and $F_1 = 1$.

Anyways, back to the Fibonacci numbers.
(a.k.a. the bee numbers)

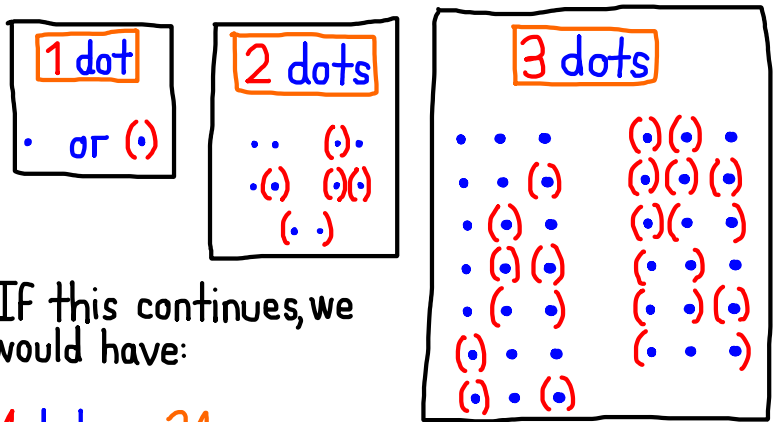
[1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, ...]

(These puzzles are from James Tanton)

In various puzzles, every 2nd Fibonacci number corresponds to the number of combinations for each instance.

Puzzle #1: Non-nested dots

In how many different ways can one place non-nested parentheses around a row of n dots?



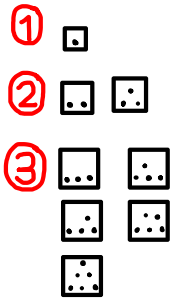
If this continues, we would have:

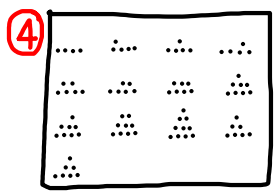
4 dots = 34 ways
5 dots = 89 ways
6 dots = 233 ways

7 dots = 610 ways
and so on!

Puzzle #2: Cannonballs

In how many ways can I stack cannon balls on a row of n balls with rows contiguous?





One row of...
5 balls → 34 ways
6 balls → 89 ways
7 balls → 233 ways
8 balls → 610 ways
and so on!

Puzzle #3: Xs and Os

In how many ways can write a string of length n **X**s and **O**s avoiding two consecutive **X**s?

Length 1: O, X

Length 2: OO, OX, XO

Length 3: OOO, OOX, OXO, XOO, XOX

Length 4: OOOO, OOOX, OOXO, OXOO, XOOO, OXOX, XOOX, XOXO

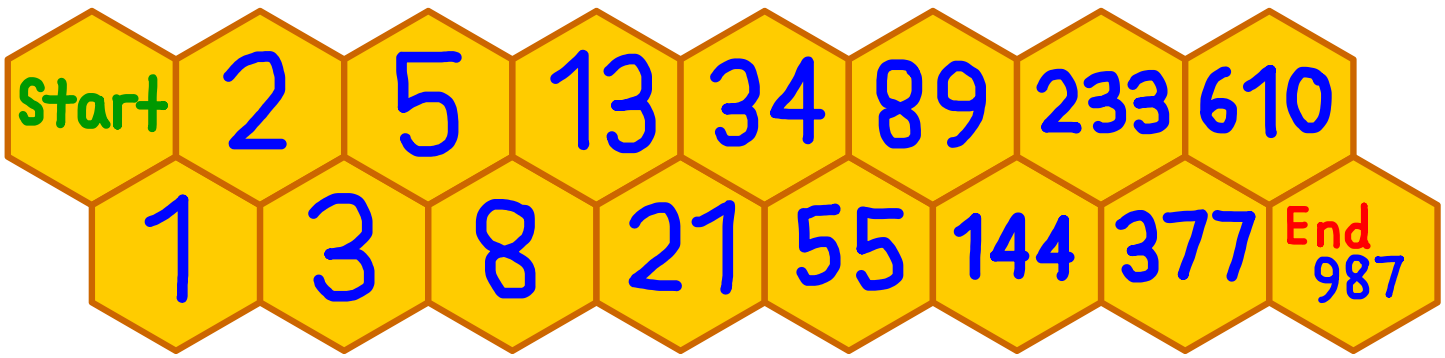
$L_5 \rightarrow 13$ ways
$L_6 \rightarrow 21$ ways
$L_7 \rightarrow 34$ ways
$L_8 \rightarrow 55$ ways
$L_9 \rightarrow 89$ ways

Unlike puzzles 1 and 2, every single Fibonacci number corresponds to the number of combination of each length of Xs and Os.

There are 4 other puzzles that Dr. Tanton did.

Puzzle #4: In how many ways can one write a number N as a sum of positive integers if there are two types of "1"?
Puzzle #5: In how many ways can one write a number N as a sum of positive integers, **avoiding** the number "1"?
Puzzle #6: Weird Products
Puzzle #7: If N is a multiple of a, then β_n is a multiple of β_a.

Now, onto the honeycombs!



You can go →, ↘ or ↗.

Finally, when we have Fibonacci numbers consisting of $\frac{F_n}{F_{n-1}}$, the quotient approaches the golden ratio where $\varphi = \frac{1+\sqrt{5}}{2} = 1.618033\ldots$

$$[1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144]$$
$$\div [1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89]$$
$$1,\ 2,\ 1.5,\ 1.\overline{6},\ 1.6,\ 1.625,\ 1.615,\ 1.619,\ 1.6176,\ 1.6182,\ 1.6180$$

Therefore,

$$\lim_{n\to\infty} \frac{F_n}{F_{n-1}} = \varphi$$